COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **19** (4) (1999) 487–505

LIST HOMOMORPHISMS AND CIRCULAR ARC GRAPHS

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Received July 22, 1996
Revised June 10, 1998

List homomorphisms generalize list colourings in the following way: Given graphs G,H, and lists $L(v) \subseteq V(H), v \in V(G)$, a *list homomorphism* of G to H *with respect to the lists* L is a mapping $f:V(G) \to V(H)$, such that $f(u)f(v) \in E(H)$ for all $uv \in E(G)$, and $f(v) \in L(v)$ for all $v \in V(G)$. The *list homomorphism problem* for a fixed graph H asks whether or not an input graph G together with lists $L(v) \subseteq V(H)$, $v \in V(G)$, admits a list homomorphism with respect to L. We have introduced the list homomorphism problem in an earlier paper, and proved there that for reflexive graphs H (that is, for graphs H in which every vertex has a loop), the problem is polynomial time solvable if H is an interval graph, and is NP-complete otherwise. Here we consider graphs H without loops, and find that the problem is closely related to circular arc graphs. We show that the list homomorphism problem is polynomial time solvable if the complement of H is a circular arc graph of clique covering number two, and is NP-complete otherwise. For the purposes of the proof we give a new characterization of circular arc graphs of clique covering number two, by the absence of a structure analogous to Gallai's asteroids. Both results point to a surprising similarity between interval graphs and the complements of circular arc graphs of clique covering number two.

1. Introduction

A *homomorphism* f of a graph G to a graph H is a mapping $f:V(G) \to V(H)$ which preserves edges, i.e., such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. If there is a homomorphism of G to H, we write $G \to H$. For a fixed graph H, the *homomorphism problem* HOMH asks whether or not an input graph G satisfies $G \to H$. When H is the complete graph K_n, a homomorphism of G to H is simply an n-colouring of G, and hence HOMH is precisely the n-colouring problem. It is well known that the n-colouring problem is polynomial time solvable when $n \leq 2$, and NP-complete when $n > 2$. The complexity of all homomorphism problems HOMH has been classified by P. Hell and J. Nešetřil [15]: HOMH is polynomial time solvable if H is a bipartite graph, and is NP-complete otherwise.

Given graphs G,H, and lists $L(v) \subseteq V(H), v \in V(G)$, a *list homomorphism* of G to H with respect to the lists L is a homomorphism f of G to H, such that $f(v) \in L(v)$ for all $v \in V(G)$. For a fixed graph H, the *list homomorphism problem* L-HOMH asks whether or not an input graph G, with lists L, admits a list homomorphism to H with respect to L. Note that HOMH is a restriction of

Mathematics Subject Classification (1991): 05C85, 05C75, 05C15; 68R10

L-HOMH, since setting each list $L(v) = V(H)$ has the effect of not having any list constraints. Thus L-HOMH is NP-complete when H is not bipartite [15]. As above, when $H = K_n$, a list homomorphism of G to H is simply a list colouring of G, and hence L-HOMH is the list colouring problem. The complexity of list colouring problems has been studied by J. Kratochvíl and Zs. Tuza [20]; among other results they proved that the problem is polynomial time solvable when each list has at most two elements, but NP-complete even if each list is restricted to have at most three elements.

Since homomorphisms generalize colourings they are sometimes referred to as H-colourings [15]. (An *H-colouring of G* is a homomorphism of G to H). In the same vein we could also call a list homomorphism of G to H a *list H-colouring* (both with respect to some L). In this terminology L-HOMH is the *list H-colouring problem.*

Recently, two of us [6] classified the complexity of L-HOMH for the case when H is a *reflexive* graph, i.e., when each vertex of H has a loop. Reflexive graphs are not interesting as far as HOMH is concerned (every graph admits a homomorphism to a reflexive H). However, list homomorphisms for reflexive graphs are both natural and interesting [6]. It turns out that L-HOMH is polynomial time solvable if H is an interval graph, and is NP-complete otherwise. Thus, in the reflexive case, the 'easy' graphs H for L-HOMH are interval graphs.

In this paper, we shall consider graphs without loops. We shall prove that L-HOMH is polynomial time solvable if the complement of H is a circular arc graph of clique covering number two, and is NP-complete otherwise. Thus, in this case, the 'easy' graphs H for L-HOMH are the complements of circular arc graphs of clique covering number two.

Our result depends on understanding the structure of circular arc graphs of clique covering number two. For this purpose, we give a new characterization of this class of graphs. Our characterization is derived from the forbidden subgraph characterization of [28], but is expressed in terms of a new forbidden structure analogous to the asteroids introduced by T. Gallai [10]. A special case of these asteroids (the so-called 'asteroidal triples') were used earlier by Lekkerkerker and Boland to give a characterization of interval graphs, and we express our characterization in a language very similar to the result of Lekkerkerker and Boland. This further underscores the similarity between interval graphs and the complements of circular arc graphs of clique covering number two.

The paper is organized as follows: In Section 2, we give a polynomial time algorithm to solve L-HOMH when the complement of H is a circular arc graph of clique covering number two. In Section 3, we show that L-HOMH is NP-complete when H contains certain structures. In Section 4, we show that the absence of these structures precisely characterizes circular arc graphs of clique covering number two. This allows us to conclude our main result, in Section 5, stating that the complements of circular arc graphs of clique covering number two are precisely the 'easy' graphs for L-HOMH. Section 5 also contains a discussion of certain modifications of the list homomorphism problem, including the more general list partition problem [8].

2. The polynomial cases of L-HOMH

A graph S is a *circular arc graph* if there is a family of arcs $A_v, v \in V(S)$, on a fixed circle, such that v and v' are adjacent in S if and only if the corresponding circular arcs A_v and $A_{v'}$ intersect. (An *interval graph* is defined analogously — each A_v being an interval.) Circular arc graphs have many applications [26], [31], and have been extensively investigated [1], [11], [12], [17], [30]. It turns out that L-HOMH is intimately related to circular arc graphs of *clique covering number two*. These are the circular arc graphs whose vertices can be covered by two cliques, i.e., whose complements are bipartite. They have received particular attention in the literature [17], [25], [28]. We shall say more about them in Section 4.

Let S be a circular arc graph whose vertices are covered by disjoint cliques X and Y. According to [25] (see also [17]), S can be represented by a family of arcs, on a circle with two specified points p and q, such that each circular arc corresponding to a vertex in X contains p but not q, and each arc corresponding to a vertex in Y contains q but not p.

We now formulate our algorithm in the language of the complement of S, which is a bipartite graph H.

Theorem 2.1. *Let H be a bipartite graph. If the complement of H is a circular arc graph, then* L-HOMH *is polynomial time solvable.*

Proof. We shall give a polynomial reduction of L-HOMH to the (polynomial time solvable) 2-satisfiability problem [4]. Let (X,Y) be a bipartition of H. We assume that the vertices of H and hence the members of all lists L of each input graph are represented by circular arcs A_v, $v \in V(H)$, such that vv' is an edge of H if and only if A_v and $A_{v'}$ do *not* intersect. We assume that the endpoints of all circular arcs are distinct and, as above, that there are two points p and q on the circle such that each circular arc corresponding to a vertex in X contains p but not q, and each arc corresponding to a vertex in Y contains q but not p. We may further assume that the input graph G is a bipartite graph with a bipartition (A,B) such that the list of each vertex of A is a subset of X and the list of each vertex of B a subset of Y. For any two points s and t on the circle, we use $[s,t]$ to denote the portion of the circle extending from s to t in the clockwise direction.

Let P denote the set of all endpoints of the circular arcs A_v, $v \in V(H)$, and let $P_{[s,t]}$ denote all points of P contained in $[s,t]$. The instance of the 2-satisfiability problem corresponding to the input graph G with lists L, will have a variable $l_{v,c}$ for each $v \in V(G)$ and each $c \in P_{[q,p]} \cup \{p,q\}$, and a variable $r_{v,c}$ for each $v \in V(G)$ and each $c \in P_{[p,q]} \cup \{p,q\}$. We shall define the clauses in such a way that they can be satisfied if and only if G admits a list homomorphism to H with respect to the lists L. We think of $l_{v,c} = 1$ or $r_{v,c} = 1$ as meaning that v is assigned a circular arc which does *not* contain the point c. (We will later explain how to use the truth values of these variables to actually choose such circular arcs.) To ensure that adjacent vertices of G are assigned non-intersecting circular arcs, we state the clauses $l_{u,c} \vee l_{v,c}$ and $r_{u,c} \vee r_{v,c}$ for each ordered pair u, v of adjacent vertices of G and each point c for which the variables are defined. To ensure that we can choose a circular arc from $L(v)$ for each vertex $v \in V(G)$, we impose the clauses $\overline{l}_{v,c} \vee \overline{r}_{v,d}$ for each $v \in V(G)$ and for each c and d for which the variables are defined and such that $[c^+, d^-]$ does not contain any circular arc from $L(v)$. (Here c^+ denote the point next to c in the clockwise direction and d^- denote the point next to to d in the counterclockwise direction.) Finally, to ensure that at least one $l_{v,c}$ and at least one $r_{v,c}$ is true for each vertex v, we impose the clauses $l_{v,q}$ and $r_{v,q}$ for $v \in A$ and the clauses $l_{v,p}$ and $r_{v,p}$ for $v \in B$. We now claim that all these clauses are satisfiable if and only if G admits a list homomorphism to H with respect to the lists L. Indeed, if such a list homomorphism f exists, we set the values $l_{v,c}$ and $r_{v,c}$ as suggested above, that is, let $l_{v,c} = 1$ and $r_{v,c} = 1$ for each c for which the variables are defined and which is not contained in the circular arc corresponding to $f(v)$. On the other hand, a satisfying truth assignment can be used to define a list homomorphism as follows: Suppose that $v \in A$. Let c be the furthest point of $P_{[q,p]} \cup \{p, q\}$ in the clockwise direction such that $l_{v,c} = 1$ and let d be the furthest point of $P_{[p,q]} \cup \{p, q\}$ in the counterclockwise direction with $r_{v,d} = 1$. Then $[c^+, d^-]$ must contain at least one circular arc from $L(v)$, and we define one such arc to be $f(v)$. When $v \in B$, $f(v)$ is defined in an analogous way. The clauses ensure that adjacent vertices are assigned non-intersecting circular arcs, i.e., adjacent vertices of H. ∎

3. The NP-complete cases of L-HOMH

In this section, we shall describe certain structures whose presence in H implies that L-HOMH is NP-complete. One family of such structures are odd cycles. Indeed, we have mentioned above that when H is nonbipartite then L-HOMH is NP-complete. Let us now consider bipartite graphs H. Even here there are cycles that make the problem hard:

Theorem 3.1. *If H contains a chordless cycle of length greater than four, then* L-HOMH *is NP-complete.*

Proof. It is clear that L-HOMH is in NP. Let $C = h_1 h_2 \ldots h_{2k} h_1$ $(k \geq 3)$ be a chordless cycle contained in H. We present a reduction from k-colourability. For any graph F, we shall construct (in polynomial time) a graph G with lists $L(v) \subseteq V(C) \subseteq V(H)$, $v \in V(G)$, in such a way that F is k-colourable if and only if G admits a list homomorphism to H with respect to the list L. First of all, the graph G contains a fixed copy C_0 of C. Next, each edge xy of F is replaced by a copy of a gadget. The gadget contains a new vertex z, a path of length $k-2$ between x and z, and two internally disjoint paths of length k between y and z. The latter two paths form a cycle C_{xy} isomorphic to C. Finally, the gadget also contains a sequence of $k-2$ new copies of C, connecting C_{xy} to C_0, in such a way that consecutive copies are joined so that the i-th vertex of one copy is adjacent to the i-th vertex of the next copy (see Figure 1). Note that except for the common cycle C_0, all vertices other than x, y are disjoint for each copy of the gadget. The lists are defined as follows: For each vertex g of C_0 we let $L(g)=\{g\}$; for all other vertices g of G we let $L(g)=V(C)$. Note that the connection between consecutive copies of C in the chain connecting each C_{xy} to C_0 assures that in any list homomorphism of G to C the i-th vertex of one copy can be only identified with the $(i-1)$-st vertex or $(i+1)$-st vertex of the next copy. This implies that each cycle must rotate one step clockwise or one step counterclockwise to the next cycle. Suppose without loss of generality that a list homomorphism f of G to the cycle C maps y to the first vertex, i.e., the vertex h_1. Then f must map z to the opposite vertex h_{k+1}. Hence f maps x to one of $h_3, h_5, \ldots, h_{2k-1}$. Moreover, for any two vertices of C with odd subscripts there is a list homomorphism of G to C that maps x and y to these two vertices. Therefore G admits a list homomorphism to C if and only if the vertices of F can be labeled by the vertices of C with odd subscripts so that adjacent vertices receive distinct labels, i.e., if and only if F is k-colourable. ∎

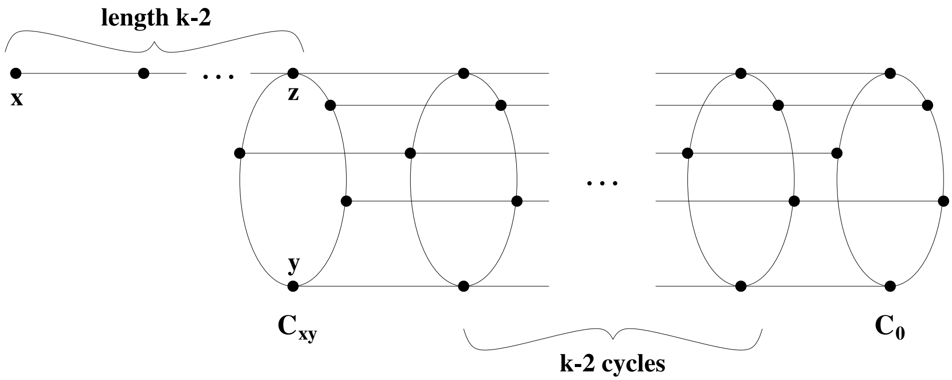

Fig. 1. Connections between C_{xy} and C_0

The last type of structures which make L-HOMH hard are defined below:

An *edge-asteroid* in a bipartite graph with the bipartition (X,Y) is a set of $2k+1$ edges $u_0v_0, u_1v_1, \ldots, u_{2k}v_{2k}$ ($k \geq 1$ and each $u_i \in X$ and $v_i \in Y$), and $2k+1$ paths, $P_{0,1}, P_{1,2}, \ldots, P_{2k,0}$, where each $P_{i,i+1}$ joins u_i to u_{i+1}, such that for each $i=0,1,\ldots,2k$ there is no edge between $\{u_i, v_i\}$ and $\{v_{i+k}, v_{i+k+1}\} \cup V(P_{i+k,i+k+1})$. (Subscripts are modulo $2k+1$.) We refer to the (odd) integer $2k+1$ as the *order* of the edge-asteroid. An edge-asteroid which has no edge between $\{u_0, v_0\}$ and $\{v_1, v_2, \ldots, v_{2k}\} \cup V(P_{1,2}) \cup V(P_{2,3}) \cup \ldots \cup V(P_{2k-1,2k})$ is called a *special edge-asteroid.*

Let $W = w_0w_1 \ldots w_k$ be a walk in a graph H. The *reverse* W^- of W is the walk $w_k \ldots w_1w_0$. With W we shall associate a path $P(W)$, which will be a graph separate from H. Specifically, an *associated path* $P(W)$ of W is a path, $p_0p_1 \ldots p_k$. Note that while the vertices of W are in H and are not necessarily distinct, the

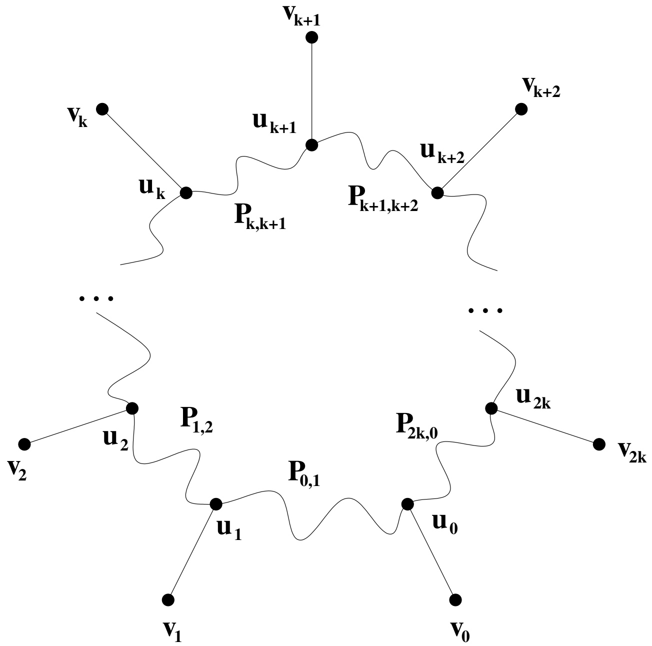

Fig. 2. An edge-asteroid

vertices of $P(W)$ are distinct, but not necessarily vertices of H. In order to be able to refer to the vertex of $P(W)$ associated with w_i we make the association specific by writing $w_i = \alpha(p_i)$. Association allows us to define a kind of concatenation of walks even if the second walk does not begin where the first walk ends: We can choose the associated paths so that the second associated path does begin where the first associated path ends, and then simply concatenate the associated paths. Specifically, we define an *A-concatenation* of two walks $W = w_0w_1 \ldots w_k$ and $W' = w'_0w'_1 \ldots w'_l$, by associating with W a path $p_0p_1 \ldots p_k$, and with W' a path $p_kp_{k+1} \ldots p_{k+l}$, and then letting the A-concatenation be defined as the *path* $p_0p_1 \ldots p_kp_{k+1} \ldots p_{k+l}$. Note that we are somewhat abusing the notation, as α is not necessarily a function — for instance the vertex p_k above has two vertices (w_k and w'_0) associated with it; nevertheless this is convenient and will not cause a problem.

Theorem 3.2. *Let H be a bipartite graph. If H contains a special edge-asteroid, then* L-HOMH *is NP-complete.*

Proof. It is again clear that L-HOMH is in NP. We shall reduce 3-colourability to L-HOMH: For any graph F we shall construct (in polynomial time) a graph G with lists $L(g) \subseteq V(H)$, $g \in V(G)$, in such a way that F is 3-colourable if and only if G admits a list homomorphism to H with respect to L. The key to the construction of F are modules called 'choosers'. A slightly different version of choosers was first introduced in [6, 16].

Fix three vertices a,b,c in H. Let A,B,C be subsets of $\{a,b,c\}$. An $(a,A;b,B;c,C)$-*chooser* is a path P with endpoints s and t, and with lists $L(p)\subseteq V(H)$, $p\in V(P)$, such that the following properties hold:

1. Any list homomorphism $f:P\to H$ with respect to L has $f(s)=a$ and $f(t)\in A$, or $f(s)=b$ and $f(t)\in B$, or $f(s)=c$ and $f(t)\in C$.
2. For any $i\in A$ (resp. $j\in B$, $k\in C$), there exists a list homomorphism $f:P\to H$ with respect to L such that $f(s)=a$ and $f(t)=i$ (resp. $f(s)=b$ and $f(t)=j$, resp. $f(s)=c$ and $f(t)=k$).

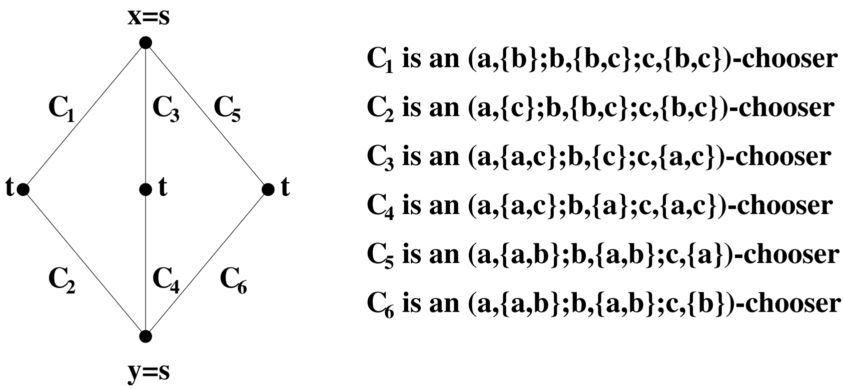

Fig. 3. The edge xy is replaced by three disjoint paths made from six choosers

In constructing the graph G from a given graph F, we replace each edge xy of F by the gadget described in Figure 3. It consists of three disjoint paths, each composed of two choosers; the 's' vertices of the choosers C_1,C_3,C_5 are identified with the vertex x, and the 's' vertices of C_2,C_4,C_6 with the vertex y. Moreover, as shown in the figure, the 't' vertices of C_1 and C_2 (respectively C_3 and C_4, respectively C_5 and C_6) are also identified. We claim that F is 3-colourable if and only if G admits a list homomorphism to H. Suppose that F is coloured by a,b,c. Let x and y be two adjacent vertices of F. Then x and y receive different colours, say a and b respectively. (The argument is similar for the other pairs a,c and b,c.) It suffices to show that there is a list homomorphism from the gadget in Figure 3 to H which maps x to a and y to b.

Let P be the path in the gadget consisting of the choosers C_1 and C_2. Since there is a list homomorphism of C_1 to H which maps s to a and t to b, and there is a list homomorphism of C_2 to H which maps s to b and t to b, there exists a list homomorphism of the path P to H which maps x to a and y to b. Similarly one can show that each of the other two paths admits a list homomorphism to H which maps x to a and y to b.

Conversely, assume that f is a list homomorphism of G to H. Let x and y be two adjacent vertices of F. We claim that f must map x and y to different vertices. Indeed, suppose that f maps both x and y to the same vertex, say a. Consider again the path P joining x and y consisting of C_1 and C_2. Then f maps the vertex

s of both choosers to a. Hence f maps the vertex t of A to b and maps the vertex t of B to c. However, the vertices t of these two choosers are identified in G, thus they cannot be mapped to two different vertices. A similar contradiction arises from assuming that f maps both x and y to b, or both x and y to c. Therefore F is 3-colourable.

It remains to construct the six choosers used in the above proof. Suppose that (X,Y) is a bipartition of H. By our assumption, H contains a special edge-asteroid with the edges $u_0v_0, u_1v_1, \ldots, u_{2k}v_{2k}$ and the paths $P_{0,1}, P_{1,2}, \ldots, P_{2k,0}$. Recall that $u_i \in X$ and $v_i \in Y$ for each $i = 0,1,\ldots,2k$. Let $a = u_0$, $b = u_1$, and $c = u_{k+1}$. We first construct an $(a,\{b\};b,\{b,c\};c,\{b,c\})$-chooser as follows: Let Q be the A-concatenation of

$$P_{1,2}, P_{2,3}, \ldots, P_{k,k+1}, P^-_{k,k+1}, P^-_{k-1,k}, \ldots, P^-_{1,2}, P_{0,1}.$$

Recall that this A-concatenation is a path, with distinct vertices. We define lists $L(q)$ for the vertices q of Q:
For each q with $\alpha(q) \in X$, we let

$$L(q) = \begin{cases} \{\alpha(q), u_0, u_{k+1}\} & \text{if } \alpha(q) \in V(P_{j,j+1}) \text{ with } j \neq 0 \\ \{\alpha(q), u_1, u_{k+1}\} & \text{if } \alpha(q) \in V(P_{0,1}) \end{cases}$$

For each q with $\alpha(q) \in Y$, we let

$$L(q) = \begin{cases} \{\alpha(q), v_0, v_{k+1}\} & \text{if } \alpha(q) \in V(P_{j,j+1}) \text{ with } j \neq 0 \\ \{\alpha(q), v_1, v_{k+1}\} & \text{if } \alpha(q) \in V(P_{0,1}) \end{cases}$$

Even though a concatenation may lead to a vertex q with two associated values of $\alpha(q)$, it is easy to verify that these definitions are not ambiguous. (In fact, only the concatenation vertex q of $P^-_{1,2}$ and $P_{0,1}$ has two associated values of $\alpha(q)$, namely, u_1 and u_0, and they result in the same $L(q)$.)

We now verify that Q together with the lists L is an $(a,\{b\};b,\{b,c\};c,\{b,c\})$-chooser. Let f be a list homomorphism of Q to H with respect to L. Assume that $f(s) = a = u_0$. The additional property of a special edge-asteroid states that there is no adjacency between $\{u_0, v_0\}$ and $\{v_1, v_2, \ldots, v_{2k}\} \cup V(P_{1,2}) \cup V(P_{2,3}) \cup \ldots \cup V(P_{2k-1,2k})$. This implies that $f(q) = u_0$ or v_0, when $\alpha(q) \in V(P_{1,2}) \cup V(P_{2,3}) \cup \ldots \cup V(P_{k,k+1}) \cup V(P^-_{k,k+1}) \cup V(P^-_{k-1,k}) \cup \ldots \cup V(P^-_{1,2})$. However there could be adjacency between $\{u_0, v_0\}$ and $V(P_{0,1})$. So, when $\alpha(q) \in V(P_{0,1})$, we have $f(q) = \alpha(q)$ or u_1 if $\alpha(q) \in X$, and $f(q) = \alpha(q)$ or v_1 if $\alpha(q) \in Y$. Since $\alpha(t) = u_1 = b$, we must have $f(t) = b$.

Suppose now that $f(s) = b = u_1$. Then, for each q with $\alpha(q) \in V(P_{1,2}) \cup V(P_{2,3}) \cup \ldots \cup V(P_{k,k+1}) \cup V(P^-_{k,k+1}) \cup V(P^-_{k-1,k}) \cup \ldots \cup V(P^-_{1,2})$, we have $f(q) = \alpha(q)$ or u_{k+1} when $\alpha(q) \in X$, and $f(q) = \alpha(q)$ or v_{k+1} when $\alpha(q) \in Y$. When $\alpha(q) \in V(P_{0,1})$, we have $f(q)=u_1$ or u_{k+1} or $\alpha(q)$ when $\alpha(q)\in X$, and $f(q)=v_1$ or v_{k+1} or $\alpha(q)$ when $\alpha(q)\in Y$. Since $\alpha(t)=u_1=b$, we must have $f(t)=u_1$ or u_{k+1}, i.e., $f(t)\in\{b,c\}$.

Finally, the case of $f(s)=c=v_{k+1}$ is similar to that of $f(s)=b=u_1$. For each q with $\alpha(q)\in V(P_{1,2})\cup V(P_{2,3})\cup\ldots\cup V(P_{k,k+1})\cup V(P^-_{k,k+1})\cup V(P^-_{k-1,k})\cup\ldots\cup V(P^-_{1,2})$, we have $f(q)=u_{k+1}$ or $\alpha(q)$ when $\alpha(q)\in X$, and $f(q)=v_{k+1}$ or $\alpha(q)$ when $\alpha(q)\in Y$. For each q with $\alpha(q)\in V(P_{0,1})$, we have $f(q)=u_{k+1}$ or v_1 or $\alpha(q)$ when $\alpha(q)\in X$, and $f(q)=v_{k+1}$ or v_1 or $\alpha(q)$ when $\alpha(q)\in Y$. Since $\alpha(t)=u_1$, we must have $f(t)=u_{k+1}$ or u_1, i.e., $f(t)\in\{b,c\}$.

Hence we have shown that the path Q together with the lists L satisfies part 1 of the definition of a chooser. Part 2 is also easy to see. In fact, when $f(s)=a$, part 1 guarantees that $f(t)=b$. When $f(s)=b$ or c, we can choose the values $f(q)$ appropriately to ensure $f(t)=b$ or to ensure $f(t)=c$. For instance, suppose that $f(s)=c=u_{k+1}$. To ensure $f(t)=b=u_1$, we can let $f(q)=u_{k+1}$ or v_{k+1} for $\alpha(q)\in V(P_{1,2})\cup V(P_{2,3})\cup\ldots\cup V(P_{k,k+1})$, let $f(q)=\alpha(q)$ for $\alpha(q)\in V(P^-_{k,k+1})\cup V(P^-_{k-1,k})\cup\ldots\cup V(P^-_{1,2})$, and let $f(q)=u_1$ or v_1 for $\alpha(q)\in V(P_{0,1})$.

Next we construct an $(a,\{c\};b,\{b,c\};c,\{b,c\})$-chooser. Let Q' be the A-concatenation of

$$P_{1,2},P_{2,3},\ldots,P_{k,k+1},$$

let Q'' be the A-concatenation of

$$P^-_{k,k+1},P^-_{2k,0},P^-_{k-1,k},P^-_{2k-1,2k},\ldots,P^-_{1,2},P^-_{k+1,k+2},$$

and let Q be the A-concatenation of Q' and Q''. To simplify the notation in defining lists $L(q)$, $q\in V(Q)$, we assume that $V(Q)=V(Q')\cup V(Q'')$, and vertices in $V(Q')$ are all distinct from vertices in $V(Q'')$ except the last vertex of Q' which is the same as the first vertex of Q''.
For each $q\in V(Q')$, let

$$L(q)=\begin{cases}\{\alpha(q),u_0,u_{k+1}\} & \text{if } \alpha(q)\in X\\ \{\alpha(q),v_0,v_{k+1}\} & \text{if } \alpha(q)\in Y\end{cases}$$

For each $q\in V(Q'')$, let

$$L(q)=\begin{cases}\{\alpha(q),u_{k+1},u_{j-k}\} & \text{if } \alpha(q)\in X\cap V(P^-_{j,j+1})\\ \{\alpha(q),v_{k+1},v_{j-k}\} & \text{if } \alpha(q)\in Y\cap V(P^-_{j,j+1})\end{cases}$$

We now verify that Q together with the lists L is an $(a,\{c\};b,\{b,c\};c,\{b,c\})$-chooser. Let f be a list homomorphism of Q to H with respect to L. Assume first that $f(s)=a=u_0$. Then $f(q)=u_0$ for $q\in V(Q')$ with $\alpha(q)\in X$ and $f(q)=v_0$ for $q\in V(Q')$ with $\alpha(q)\in Y$. When $q\in V(Q'')$, we have $f(q)=u_{k+1}$ or $\alpha(q)$ for $\alpha(q)\in X\cap V(P^-_{j,j+1})$

with $j=2k,2k-1,\dots,k+1$, and $f(q)=u_{k+1}$ or u_{j-k} for $\alpha(q)\in X\cap V(P^-_{j,j+1})$ with $j=k,k-1,\dots,1$. We also have $f(q)=v_{k+1}$ or $\alpha(q)$ for $\alpha(q)\in Y\cap V(P^-_{j,j+1})$ with $j=2k,2k-1,\dots,k+1$, and $f(q)=v_{k+1}$ or v_{j-k} for $\alpha(q)\in Y\cap V(P^-_{j,j+1})$ with $j=k,k-1,\dots,1$. Since $\alpha(t)=u_{k+1}$, we have $f(t)=\alpha(t)=u_{k+1}=c$.

Assume now that $f(s)=b=u_1$. When $q\in V(Q')$, we have $f(q)=\alpha(q)$ or u_{k+1} for $\alpha(q)\in X$ and $f(q)=\alpha(q)$ or v_{k+1} for $\alpha(q)\in Y$. When $q\in V(Q'')$, we have $f(q)=u_{k+1}$ or $\alpha(q)$ for $\alpha(q)\in X\cap V(P^-_{j,j+1})$ with $j=k,k-1,\dots,1$ and $f(q)=v_{k+1}$ or $\alpha(q)$ for $\alpha(q)\in Y\cap V(P^-_{j,j+1})$ with $j=k,k-1,\dots,1$; $f(q)=u_{k+1}$ or u_{j-k} for $\alpha(q)\in X\cap V(P^-_{j,j+1})$ with $j=2k,2k-1,\dots,k+1$ and $f(q)=v_{k+1}$ or v_{j-k} for $\alpha(q)\in Y\cap V(P^-_{j,j+1})$ with $j=2k,2k-1,\dots,1$. Since $\alpha(t)=u_{k+1}$, we have $f(t)=u_{k+1}=c$ or $u_{k+1-k}=u_1=b$, i.e., $f(t)\in\{b,c\}$.

A similar proof shows if $f(s)=c=u_{k+1}$, then $f(t)\in\{b,c\}$. So this verifies part 1 of the definition of an $(a,\{c\};b,\{b,c\};c,\{b,c\})$-chooser. Part 2 is also easy to see. In fact, when $f(s)=a$, part 1 guarantees that $f(t)=c$. When $f(s)=b$ or c, we can choose the values $f(q)$ appropriately to ensure $f(t)=b$ or to ensure $f(t)=c$.

Finally, we show how to construct the four other choosers; we will omit all verifications, which can be done as above.

To construct an $(a,\{a,c\};b,\{a\};c,\{a,c\})$-chooser, we let Q be the A-concatenation of

$$P_{k+1,k+2},P_{1,2},P_{k+2,k+3},P_{2,3},\dots,P_{2k,0},P^-_{k-1,k},$$
$$P^-_{2k-1,2k},P^-_{k-2,k-1},\dots,P^-_{1,2},P^-_{k+1,k+2},P^-_{0,1}.$$

For each q with $\alpha(q)\in X\cap V(P_{j,j+1})$, let $L(q)=\{\alpha(q),u_0,u_{j-k}\}$. For each q with $\alpha(q)\in Y\cap V(P_{j,j+1})$, let $L(q)=\{\alpha(q),v_0,v_{j-k}\}$.

To construct an $(a,\{a,c\};b,\{c\};c,\{a,c\})$-chooser, we let Q' be the A-concatenation of

$$P_{k+1,k+2},P_{1,2},P_{k+2,k+3},P_{2,3},\dots,P_{2k,0},P^-_{k-1,k},$$
$$P^-_{2k-1,2k},P^-_{k-2,k-1},\dots,P^-_{1,2},P^-_{k+1,k+2},$$

let Q'' be the A-concatenation of $P_{1,2},P_{2,3},\dots,P_{k,k+1}$, and let Q be the A-concatenation of Q' and Q''. For each $q\in V(Q')$, define $L(q)=\{\alpha(q),u_0,u_{j-k}\}$ if $\alpha(q)\in X\cap V(P_{j,j+1})$, and $L(q)=\{\alpha(q),v_0,v_{j-k}\}$ if $\alpha(q)\in Y\cap V(P_{j,j+1})$. For each $q\in V(Q'')$, define $L(q)=\{\alpha(q),u_0,u_{k+1}\}$ if $\alpha(q)\in X$, and $L(q)=\{\alpha(q),v_0,v_{k+1}\}$ if $\alpha(q)\in Y$.

To construct an $(a,\{a,b\};b,\{a,b\};c,\{a\})$-chooser, we let Q' be the A-concatenation of $P_{0,1},P^-_{0,1}$, let Q'' be the A-concatenation of

$$P_{k+1,k+2},P_{1,2},P_{k+2,k+3},P_{2,3},\dots,P_{k-1,k},P_{2k,0},P^-_{k-1,k},P^-_{k-2,k-1},\dots,P^-_{1,2},$$

and let Q be the A-concatenation of Q' and Q''. For each $q \in V(Q')$, define $L(q) = \{\alpha(q), u_0, u_{k+1}\}$ if $\alpha(q) \in X$, and $L(q) = \{\alpha(q), v_0, v_{k+1}\}$ if $\alpha(q) \in Y$. For each $q \in V(Q'')$, define $L(q) = \{\alpha(q), u_0, u_{j-k}\}$ if $\alpha(q) \in X \cap V(P_{j,j+1})$, and $L(q)=\{\alpha(q), v_0, v_{j-k}\}$ if $\alpha(q) \in Y \cap V(P_{j,j+1})$.

To construct an $(a,\{a,b\};b,\{a,b\};c,\{b\})$-chooser, we let Q' be the A-concatenation of $P_{0,1}, P^-_{0,1}$, let Q'' be the A-concatenation of $P^-_{k,k+1}, P^-_{k-1,k}, \ldots, P^-_{1,2}$, and let Q be the A-concatenation of Q' and Q''. For each $q \in V(Q')$, define $L(q) = \{\alpha(q), u_0, u_{k+1}\}$ if $\alpha(q) \in X$, and $L(q) = \{\alpha(q), v_0, v_{k+1}\}$ if $\alpha(q) \in Y$. For each $q \in V(Q'')$, define $L(q)=\{\alpha(q), u_0, u_1\}$ if $\alpha(q) \in X$, and $L(q)=\{\alpha(q), v_0, v_1\}$ if $\alpha(q) \in Y$. ∎

Next, we shall bring these last two sections together by showing that the structures making L-HOMH difficult are precisely the forbidden subgraphs for circular arc graphs of clique covering number two.

4. Circular arc graphs of clique covering number two

Circular arc graphs of clique covering number two have been characterized in various ways. J. Spinrad [25] gave a characterization in terms of the dimension of an associated poset. A simpler characterization was recently given by P. Hell and J. Huang [17]:

Let S be a graph which can be covered by two cliques. We define the auxiliary graph S^* of S as follows: The vertex set is $V(S^*) = E(S)$, and two edges of S are adjacent in S^* if their endpoints induce a chordless four-cycle in S.

Theorem 4.1. ([17]) *Let S be a graph which can be covered by two cliques. Then S is a circular arc graph if and only if the auxiliary graph S^* is bipartite.* ∎

A forbidden subgraph characterization of circular arc graphs of clique covering number two was found by Trotter and Moore [28]. It is customary to state the Trotter-Moore characterization in terms of the complementary bipartite graphs. To concisely describe their forbidden (bipartite) subgraphs, Trotter and Moore employ the following notation: Let $\mathcal{F} = \{T_i : 1 \le i \le k\}$ be a family of subsets of $\{1,2,\ldots,l\}$. Define $H_{\mathcal{F}}$ to be the bipartite graph (X,Y) with $X = \{x_1, x_2, \ldots, x_l\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ such that $x_i y_j$ is an edge if and only if $i \in T_j$.

Theorem 4.2. *A bipartite graph H is the complement of a circular arc graph if and only if H contains no $H_{\mathcal{F}}$ for any $\mathcal{F}$ in the table shown below.* ∎

$$\mathcal{C}_3 = \{\{1,2\},\{2,3\},\{3,1\}\}$$
$$\mathcal{C}_4 = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$$
$$\mathcal{C}_5 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$$
$$\ldots$$
$$\mathcal{T}_1 = \{\{1,2\},\{2,3\},\{3,4\},\{2,3,5\},\{5\}\}$$
$$\mathcal{T}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{2,3,4,6\}\{6\}\}$$
$$\mathcal{T}_3 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{2,3,4,5,7\},\{7\}\}$$
$$\ldots$$
$$\mathcal{W}_1 = \{\{1,2\},\{2,3\},\{1,2,4\},\{2,3,4\},\{4\}\}$$
$$\mathcal{W}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3,5\},\{2,3,4,5\},\{5\}\}$$
$$\mathcal{W}_3 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,2,3,4,6\},\{2,3,4,5,6\},\{6\}\}$$
$$\ldots$$
$$\mathcal{D}_1 = \{\{1,2,5\},\{2,3,5\},\{3\},\{4,5\},\{2,3,4,5\}\}$$
$$\mathcal{D}_2 = \{\{1,2,6\},\{2,3,6\},\{3,4,6\},\{4\},\{5,6\},\{2,3,4,5,6\}\}$$
$$\mathcal{D}_3 = \{\{1,2,7\},\{2,3,7\},\{3,4,7\},\{4,5,7\},\{5\},\{6,7\},\{2,3,4,5,6,7\}\}$$
$$\ldots$$
$$\mathcal{M}_1 = \{\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,4,6\},\{2,4\},\{2,5\}\}$$
$$\mathcal{M}_2 = \{\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,7\}\}$$
$$\mathcal{M}_3 = \{\{1,2,3,4,5,6,7,8,9\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,5,6,8,10\},$$
$$\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,9\}\}$$
$$\ldots$$
$$\mathcal{N}_1 = \{\{1,2,3\},\{1\},\{1,2,4,6\},\{2,4\},\{2,5\},\{6\}\}$$
$$\mathcal{N}_2 = \{\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,7\},\{8\}\}$$
$$\mathcal{N}_3 = \{\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,5,6,8,10\},\{1,2,3,4,6,8\},$$
$$\{1,2,4,6\},\{2,4\},\{2,9\},\{10\}\}$$
$$\ldots$$
$$\mathcal{G}_1 = \{\{1,3,5\},\{1,2\},\{3,4\},\{5,6\}\}$$
$$\mathcal{G}_2 = \{\{1\},\{1,2,3,4\},\{2,4,5\},\{2,3,6\}\}$$
$$\mathcal{G}_3 = \{\{1,2\},\{3,4\},\{5\},\{1,2,3\},\{1,3,5\}\}$$

(The above families are grouped according to common types — we shall refer to each family $\mathcal{C}_i$ as a family of *type* $\mathcal{C}$, each family $\mathcal{T}_i$ as a family of *type* $\mathcal{T}$, etc.)

This characterization is somewhat unwieldy for our purposes, and we shall simplify it in terms of edge-asteroids, introduced in the previous section.

A bipartite graph H is *chordal bipartite* if it contains no chordless cycle of length at least 6.

Lemma 4.3. *If H is a bipartite graph and the complement of H is a circular arc graph, then H is chordal bipartite and contains no edge-asteroids.*

Proof. If H is the complement of a circular arc graph, then Theorem 4.2 assures that H contains no $H_{\mathcal{F}}$ for any family $\mathcal{F}$ of type $\mathcal{C}$ (from the table). Those $H_{\mathcal{F}}$ are precisely the chordless cycles of even lengths at least 6. Hence H is chordal bipartite.

On the other hand, we shall show that if H contains an edge-asteroid then the complement S of H is not a circular arc graph. According to Theorem 4.1, it suffices to show that the auxiliary graph S^* of S contains a cycle of odd length. So consider an edge-asteroid in H which consists of $2k+1$ edges $u_0v_0, u_1v_1, \ldots, u_{2k}v_{2k}$ and $2k+1$ paths $P_{0,1}, P_{1,2}, \ldots, P_{2k,0}$. Let $P_{i+k,i+k+1} = u_{i+k}, p_1, p_2, \ldots, p_{2l+1}, u_{i+k+1}$. Since there is no edge between $\{u_i, v_i\}$ and $\{v_{i+k}, v_{i+k+1}\} \cup V(P_{i+k,i+k+1})$, there is, in S^*, a walk $u_iv_{i+k}, u_{i+k}v_i, u_ip_1, p_2v_i, \ldots, u_ip_{2l+1}, u_{i+k+1}v_i$. That is, S^* contains a walk from u_0v_k to $u_{k+1}v_0$, a walk from $u_{k+1}v_0$ to u_1v_{k+1}, a walk from u_1v_{k+1} to $u_{k+2}v_1$, $\ldots$, a walk from u_kv_{2k} to u_0v_k. Clearly these walks form a closed walk. Since each of the walks is of odd length and the total number of walks is odd, the resulting closed walk is of odd length and hence S^* contains a cycle of odd length. ∎

Lemma 4.4. *If H is chordal bipartite and contains no edge-asteroids, then the complement of H is a circular arc graph.*

Proof. By Theorem 4.2, it suffices to show that H contains no $H_{\mathcal{F}}$ for any $\mathcal{F}$ in the table. Since H is chordal bipartite, it does not contain any $H_{\mathcal{F}}$ with $\mathcal{F}$ being a family of type $\mathcal{C}$. So assume that H contains $H_{\mathcal{F}}$ with $\mathcal{F}$ being of type $\mathcal{T}, \mathcal{W}, \mathcal{D}, \mathcal{M}, \mathcal{N}$, or $\mathcal{G}$. We shall show that H contains an edge-asteroid. The definition of an edge-asteroid requires us to identify a set of edges $u_0v_0, u_1v_1, \ldots, u_{2k}v_{2k}$, and a set of paths $P_{0,1}, P_{1,2}, \ldots, P_{2k,0}$, and verify the corresponding properties. In most cases it will be sufficient to specify the edges $u_0v_0, u_1v_1, \ldots, u_{2k}v_{2k}$; it is then quite easy to find paths which satisfy the required properties. In other cases we will actually specify the paths $P_{0,1}, P_{1,2}, \ldots, P_{2k,0}$, and sketch how to verify the properties.

For $\mathcal{F}$ of most types, the graph $H_{\mathcal{F}}$ actually contains an edge-asteroid of order 3. We will begin with these families, starting with the finitely many families of type $\mathcal{G}$. Thus let $\mathcal{F} = \mathcal{G}_i$. If $i = 1$, then we take the three edges x_2y_2, x_3y_4, x_4y_6; if $i = 2$, we take the three edges x_1y_1, x_3y_5, x_4y_6; if $i = 3$, we take the three edges x_1y_1, x_2y_4, x_3y_5.

For the infinite groups of families, in the spirit of [28], we only give a careful definition for the first few members of the family. (The general definition follows the pattern of these cases.) If $\mathcal{F} = \mathcal{T}_i$, then we use x_1y_1, x_3y_4, x_5y_5 for $i = 1$, x_1y_1, x_4y_5, x_6y_6 for $i = 2$, x_1y_1, x_5y_6, x_7y_7 for $i = 3$, and so on. If $\mathcal{F} = \mathcal{W}_i$, we use x_1y_1, x_2y_3, x_5y_4 for $i = 1$, x_1y_1, x_3y_4, x_6y_5 for $i = 2$, x_1y_1, x_4y_5, x_7y_6 for $i = 3$, and so on. If $\mathcal{F} = \mathcal{D}_i$, we use x_1y_1, x_3y_3, x_4y_4 for $i = 1$, x_1y_1, x_4y_4, x_5y_5 for $i = 2$, x_1y_1, x_5y_5, x_6y_6 for $i = 3$, and so on.

For the last two types of families, $\mathcal{M}, \mathcal{N}$, the edge-asteroids will have higher orders. Let $\mathcal{F} = \mathcal{M}_i$. If $i = 1$, then $H_{\mathcal{F}}$ contains an edge-asteroid of order 5 with the edges x_6y_5, x_5y_4, x_4y_6, x_3y_1, x_2y_3; if $i = 2$, $H_{\mathcal{F}}$ contains an edge-asteroid of order 7 with the edges x_8y_7, x_7y_4, x_6y_6, $x_5y_8, x_4y_1, x_3y_3, x_2y_5$; if $i = 3$, $H_{\mathcal{F}}$ contains an edge-asteroid of order 9 with the edges $x_{10}y_9, x_9y_4, x_8y_6, x_7y_8, x_6y_{10}, x_5y_1, x_4y_3, x_3y_5, x_2y_7$; and so on. For these edge-asteroids of higher orders, finding suitable paths $P_{i,i+1}$ is less trivial. We shall describe the paths only for the last case ($i=3$), as the pattern will be sufficiently clear from this one case. For convenience, we number the sets in the above description of $\mathcal{M}_3$ as $T_1, T_2, \dots, T_{10}$. Observe that $T_9 \cap T_8$ contains the element 4, which is absent from T_4. Thus we can define the path $P_{1,2}$ as $x_9y_4x_8$, since it joins $u_1 = x_9$ to $u_2 = x_8$ and has no adjacencies to $u_6 = x_4, v_6 = y_3$. Similarly, the observation that $T_8 \cap T_7$ contains the element 6, which is absent from T_3, is sufficient to define the path $P_{2,3}$, the fact that $T_7 \cap T_6$ contains the element 8, which is absent from T_2, can be used to define $P_{3,4}$, the fact that $T_6 \cap T_5$ contains the element 1, which is absent from T_{10}, can be used to define $P_{4,5}$, the fact that $T_5 \cap T_4$ contains the element 1, which is absent from T_9, can be used to define $P_{5,6}$, the fact that $T_4 \cap T_3$ contains the element 3, which is absent from T_8, can be used to define $P_{6,7}$, and the fact that $T_3 \cap T_2$ contains the element 5, which is absent from T_7, can be used to define $P_{7,8}$. Thus it remains only to specify $P_{0,1}$ and $P_{8,0}$; we let $P_{0,1} = x_{10}y_2x_9$ and $P_{8,0} = x_2y_7x_0y_9x_{10}$.

The situation for the family of type $\mathcal{N}$ is similar, and we shall only specify an edge-asteroid of order 9 in $H_{\mathcal{N}_3}$: We take the edges

$$x_{10}y_{10}, x_4y_1, x_3y_3, x_2y_5, x_1y_7, x_9y_9, x_8y_4, x_7y_6, x_6y_8.$$

As above, the path $P_{1,2}$ is defined from the observation that $T_4 \cap T_3$ contains the element 1 which is absent from T_8, the path $P_{2,3}$ from the fact that $T_3 \cap T_2$ contains the element 3 which is absent from T_7, the path $P_{3,4}$ from the fact that $T_2 \cap T_1$ contains the element 5 which is absent from T_6, the path $P_{4,5}$ from the fact that $T_1 \cap T_9$ contains the element 2 which is absent from T_{10}, the path $P_{5,6}$ from the fact that $T_9 \cap T_8$ contains the element 2 which is absent from T_4, the path $P_{6,7}$ from the fact that $T_8 \cap T_7$ contains the element 4 which is absent from T_3, and the path $P_{7,8}$ from the fact that $T_7 \cap T_6$ contains the element 6 which is absent from T_2. The remaining two paths can be defined as $P_{0,1} = x_{10}y_{10}x_5y_1x_4$ and $P_{8,0} = x_6y_8x_5y_{10}x_{10}$. ∎

Combining the above two lemmas, we obtain the following theorem.

Theorem 4.5. *A bipartite graph H is the complement of a circular arc graph if and only if H is chordal bipartite and contains no edge-asteroids.* ∎

In other words, *a graph S is a circular arc graph of clique covering number two if and only if the complement of S is chordal bipartite and contains no edge-asteroids.* We observe that this is very similar to the characterization of interval graphs due to Lekkerkerker and Boland [21]: *A graph is an interval graph if and only if it is chordal and contains no asteroids.*

(The usual formulation of this characterization replaces 'asteroids' by the more specific 'asteroidal triples'; we wished to emphasize the similarity with Theorem 4.5.)

We should point out that all the edge-asteroids we defined in the proof of Lemma 4.4 are special edge-asteroids. Thus we have:

Corollary 4.6. *A bipartite graph H is the complement of a circular arc graph if and only if H is chordal bipartite and contains no special edge-asteroids.* ∎

5. Conclusions and further work

We begin by stating our main theorem of this paper, which combines Theorems 2.1, 3.1, 3.2, Corollary 4.6, and the remark at the beginning of Section 3.

Theorem 5.1. L-HOMH *is*

- *polynomial time solvable if the complement of H is a circular arc graph of clique covering number two;*
- NP-*complete otherwise.* ∎

If the lists of the input graph are restricted to induce connected subgraphs of H we have the *connected* list homomorphism problem CL-HOMH. If the lists are restricted to induce complete subgraphs of H we have the *complete* list homomorphism problem KL-HOMH. We first make some observations about CL-HOMH problems, and give a complete classification of the complexity of KL-HOMH problems.

For reflexive graphs H, it is shown in [6] that CL-HOMH is polynomial time solvable if H is a chordal graph, and is NP-complete otherwise. The obvious straightforward analogue of this statement for graphs without loops is not true. Specifically, there exist chordal bipartite graphs H for which the problem CL-HOMH is NP-complete: Consider $H = H_{\mathcal{G}_1}$ and let (X,Y) be the bipartition of H. Recall that L-HOMH is NP-complete. Now let H' be the graph obtained from H by adjoining two new vertices x and y, and all the edges between x and $Y \cup \{y\}$ and between y and X. The resulting graph H' is easily seen to be chordal bipartite. In the proof of Theorem 3.2, we constructed for every graph F, a graph G with lists $L(v) \subseteq V(H)$, $v \in V(G)$, such that F is colourable by three colours if and only if G admits a list homomorphism with respect to the lists L. It is sufficient to do this construction for connected graphs F (as it is NP-complete to decide whether or not a connected graph is 3-colourable). For connected graphs F, the corresponding graph G is also connected, according to the construction. Now we modify the graph G and its lists L as follows: Let G' be the graph obtained from G by adjoining a new vertex g and an edge between g and a vertex, say u, of G. Assume that $L(u) \subseteq Y$. Let $L'(g) = \{x\}$, let $L'(v) = L(v) \cup \{x\}$ for $v \in V(G)$ with $L(v) \subseteq Y$, and let $L'(v) = L(v) \cup \{y\}$ for $v \in V(G)$ with $L(v) \subseteq X$. Clearly the lists L' are all connected. Note that every list homomorphism f from G' to H maps g to x. Thus f must map the vertex u into $L(u)$. Since G is connected, f must map each vertex v of $V(G)$ into $L(v)$. Hence G' admits a list homomorphism to H' with respect to the lists L' if and only if G admits a list homomorphism with respect to the lists L, and if and only if F is colourable by three colours. This shows that CL-HOMH' is NP-complete.

For the complete list homomorphism problems KL-HOMH, we have the following:

Theorem 5.2. KL-HOMH *is polynomial time solvable if* H *does not contain a triangle and is NP-complete otherwise.*

Proof. If H contains the triangle a, b, c, then a given graph G is 3-colourable if and only if it has a list homomorphism to H with respect to the lists $L(v) = \{a, b, c\}$ for all v. Thus KL-HOMH is NP-complete. If H contains no triangle, then an instance of KL-HOMH is a graph G with each list consisting of either a single vertex t_v or two vertices t_v, f_v. We reduce KL-HOMH to the (polynomial time solvable) problem of 2-satisfiability: The set of variables will be $V(G)$. The set $\mathcal{C}$ of clauses will consist of u for each u with a single vertex in $L(u)$, and $u \vee v$ (respectively $\overline{u} \vee v$, $\overline{u} \vee \overline{v}$) for each ordered pair of adjacent u, v with $f_u f_v \notin E(H)$ (respectively $t_u f_v \notin E(H)$, $t_u t_v \notin E(H)$). It is easy to see that G admits a list homomorphism to H with respect to the lists L if and only if the clauses are satisfiable. ∎

The above proof actually shows that *if each list contains at most two vertices then the list homomorphism problem L-HOMH is polynomial time solvable.* This generalizes the result of Kratochvíl and Tuza (which treats the case $H = K_n$) mentioned in the introduction [20].

If the lists are restricted to consist of either a single vertex or all vertices of H, we have the so called *one-or-all* list homomorphism problem, OAL-HOMH [6]. The proof of Theorem 3.1 actually shows OAL-HOMH is NP-complete if H is a cycle of length at least 6. This was also proved by G. MacGillivray (personal communication). The problem OAL-HOMH has previously been studied as *the retraction problem* [13], [14], [18], [23]: In the retraction problem RETH, for a fixed H, we ask whether or not an input graph G which contains an isomorphic copy H' of H admits a homomorphism r to H' such that $r(x) = x$ for all vertices x in H'. (The easy fact that, for every graph H, the problems RETH and OAL-HOMH are polynomial time equivalent, is observed in [6].) The problem OAL-HOMK_n has been also studied, under the name *precolouring extension* [19]. The authors give (among other results) a polynomial time algorithm for the problem OAL-HOMK_n if the input is restricted to perfect graphs in which all singleton lists consist of the same vertex of K_n.

Finally we remark that the retract problem RETH (or the one-or-all problem OAL-HOMH) is not likely to admit a simple classification into the polynomial time solvable and the NP-complete cases. In fact, according to [9], for every constraint satisfaction problem there exists a polynomial time equivalent retraction problem. It is not even known [9] that all constraint satisfaction problems, and hence all problems RETH, are polynomial time solvable or NP-complete. Thus a detailed classification according to complexity would be quite surprising. We only observe here that there are graphs H for which RETH is polynomial time solvable but L-HOMH is NP-complete: For instance, we can take the graph H to be $H_{\mathcal{G}_1}$ or $H_{\mathcal{M}_1}$.

The list homomorphism problem to general graphs (each vertex may have a loop or not) is more difficult. We do not have a complete classification of complexity of L-HOMH problems, but we do have a conjecture [7], which states that graphs H which admit a certain geometric representation (by 'bi-arcs') give polynomial time solvable L-HOMH. If true, this conjecture would solve the classification problem, as we prove in [7] that graphs H which do not admit such a representation yield NP-complete L-HOMH. We have verified the conjecture (and hence obtained a classification) for trees H. Let V_L denote the set of vertices of H which have loops. We also prove in [7] that if H is connected but V_L is not, then RETH (and hence CL-HOMH and L-HOMH) is NP-complete. If H is a tree and V_L is connected, then CL-HOMH (and hence also RETH) is polynomial time solvable.

Assume now that M is a fixed symmetric k by k matrix with entries $0,1,2$. Given an input graph G with lists $L(v) \subseteq \{1,2,\ldots,k\}, v \in V(G)$, a *list M-partition* of *G with respect to the lists* L is a partition of the vertices of G into sets $S_1, S_2, \ldots, S_k$ such that

1. if $M(i,i)=0$ then S_i is independent
2. if $M(i,i)=2$ then S_i is a clique
3. if $M(i,j)=0$ then there are no edges between S_i and S_j
4. if $M(i,j)=2$ then every vertex of S_i is adjacent to every vertex of S_j.

It is easy to see that when M has only entries $0,1$, a list M-partition of G is a list homomorphism of G to the graph whose adjacency matrix is M (with the same lists). Similarly, if M has only entries $1,2$, a list M-partition of G is a list homomorphism of the complement of G to the graph whose adjacency matrix is $2J-M$ (J is the all-one matrix, i.e., $2J-M$ has zeros where M had twos and vice versa). Thus list partitions are in some sense list homomorphisms 'symmetrized' with respect to taking complements. The *list partition problem* L-PARTM asks whether or not a given input graph G with lists L admits a list M-partition. Many well known combinatorial problems are special cases of L-PARTM, including recognizing split graphs and their generalizations [2], the existence of clique cutsets [32], [27], independent cutsets [29], [5], homogeneous sets [22], and skew cutsets [3]. In [8] we study the complexity of the list partition problems. We give polynomial time algorithms for certain types of matrices M, and prove NP-completeness for certain other types of matrices. These results are sufficient to fully classify the complexity, as NP-complete or polynomial time solvable, for matrices of size three or less. We also give subexponential algorithms for many types of matrices M, strongly suggesting they are not NP-complete. This allows us to classify all problems for matrices of size four, as NP-complete or subexponential. In particular, we obtain the first subexponential algorithm for the skew cutset problem of Chvátal [3].

Our conjecture on the complexity of general list homomorphism problems [7], if true, would imply 'dichotomy' (NP-complete or polynomial time solvable) for list homomorphism problems. (Proving such dichotomies is rare [9], [24], [15], although there is a general conjecture [9] that there is such dichotomy for constraint satisfaction problems, of which list homomorphisms — but not list partitions — are a special case.) We have also shown [8] that if there was dichotomy for list homomorphism problems (as would follow both from the conjecture of [9] and of [7]), then every list partition problem would be NP-complete or subexponential.

References

[1] B. Bhattacharya, P. Hell, and J. Huang: A linear algorithm for maximum weighted cliques in proper circular arc graphs, *SIAM J. Disc. Math.,* **9** (1996), 274–289.

[2] A. Brandstadt: Partitions of graphs into one or two independent sets and cliques, *Discrete Mathematics,* **152** (1996), 47–54.

[3] V. Chvátal: Star-cutsets and perfect graphs, *J. Combin. Theory Ser. B,* **39** (1985), 189–199.

[4] S. A. Cook: The complexity of theorem-proving procedures, *Proc. 3rd Annual ACM Symposium on Theory of Computing,* (1971), 151–158.

[5] C. M. H. de Figueiredo and S. Klein: The NP-completeness of multipartite cutset testing, *Congressus Numerantium,* **119** (1996), 217–222.

[6] T. Feder and P. Hell: List homomorphisms to reflexive graphs, *J. Combin. Theory Ser. B,* **72** (1998), 236–250.

[7] T. Feder, P. Hell, and J. Huang: List homomorphisms to general graphs, manuscript.

[8] T. Feder, P. Hell, S. Klein, and R. Motwani: Complexity of graph partition problems, *Proc. 31st Annual ACM Symposium on Theory of Computing,* 464–472, Atlanta, GA, May 1999.

[9] T. Feder and M. Y. Vardi: Monotone monadic SNP and constraint satisfaction, *Proc. 25th Annual ACM Symposium on Theory of Computing* (1993), 612–622.

[10] T. Gallai: Transitiv orientierbare graphen, *Acta Math. Acad. Sci. Hungar.,* **18** (1967), 25–66.

[11] F. Gavril: Algorithms on circular-arc graphs, *Networks,* **4** (1974), 357–369.

[12] M. C. Golumbic: *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.

[13] P. Hell: Rétractions de graphes, Ph.D. Thesis, Université de Montréal, 1972.

[14] P. Hell: Retracts in graphs, in: *Graphs and Combinatorics*, Springer-Verlag Lecture Notes in Mathematics 406 (1974), 291-301.

[15] P. Hell and J. Nešetřil: On the complexity of H-colouring, *J. Combin. Theory Ser. B,* **48** (1990), 92–110.

[16] P. Hell, J. Nešetřil, and X. Zhu: Complexity of tree homomorphisms, *Discrete Applied Math.,* **70** (1996), 23–36.

[17] P. Hell and J. Huang: Two remarks on circular arc graphs, *Graphs and Combinatorics,* **13** (1997), 65–72.

[18] P. Hell and I. Rival: Absolute retracts in graphs, *Canad. J. Math.,* **39** (1987), 544-567.

[19] J. Kratochvíl and A. Sebő: Coloring precolored perfect graphs, *J. Graph Theory,* **25** (1997), 207–215.

[20] J. Kratochvíl and Zs. Tuza: Algorithmic complexity of list colorings, *Discrete Applied Math.,* **50** (1994), 297–302.

[21] C. G. Lekkerkerker and J. Ch. Boland: Representation of a finite graph by a set of intervals on the real line, *Fund. Math.,* **51** (1962), 45–64.

[22] R. M. McConnel and J. P. Spinrad: Linear-time modular decomposition and efficient transitive orientation of comparability graphs, *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms* 1994, pp. 536–545.

[23] E. Pesch: *Retracts of Graphs*, Athenaeum, Frankfurt am Main, 1988.

[24] T. J. Schaefer: The complexity of satisfiability problems, *Proc. 10th Annual ACM Symposium on Theory of Computing,* (1978), 216–226.

[25] J. Spinrad: Circular-arc graphs with clique cover number two, *J. Combin. Theory Ser. B,* (1988), 300–306.

[26] K. E. Stouffers: Scheduling of traffic lights—a new approach, *Transportation Res.,* **2** (1968), 199–234.

[27] R. E. Tarjan: Decomposition by clique separators, *Discrete Mathematics,* **55** (1985), 221–232.

[28] W. T. Trotter and J. I. Moore: Characterization problems for graphs, partially ordered sets, lattices, and families of sets, *Discrete Math.,* **16** (1976), 361–381.

[29] A. Tucker: Coloring graphs with stable sets, *J. Combin. Theory Ser. B,* **34** (1983), 258–267.

[30] A. Tucker: Matrix characterization of circular arc graphs, *Pacific J. Math,* **39** (2) (1971), 535–545.

[31] A. Tucker: Colouring a family of circular arcs, *SIAM J. Appl. Math.,* **29** (1975), 493–502.

[32] S. Whitesides: An algorithm for finding clique cutsets, *Information Processing Letters,* **12** (1981), 31–32.

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